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Abstract. Several authors argued that systems in entangled states are examples of emergent systems. In this paper, we challenge this point of view. We propose a criterion to distinguish composed systems and non-composed systems, and show that systems in entangled states don't verify this criterion. Because they are not composed, they cannot be examples of emergent systems.

Contents

1. Introduction

Quantum systems are often presented as examples of emergent situations, through their behaviour with respect to the property of decomposition into sub-systems. See for example [1], [9], [7], [2], [5], [6] and [4].

One encounters frequently, in the literature concerning emergence and quantum mechanics, the fact that the properties of the whole are emergent, in the sense that they cannot be reduced to the properties of the parts. The key ingredient here is the concept of entanglement. The fact that entangled states cannot be factorized would be a trace of emergence: it is impossible to attribute any property to the parts, so it is impossible to deduce the properties of the whole form the ones of the parts.

The aim of this article is first to address the question of knowing if a formalization of the key concepts involved in emergence and reduction can be achieved strictly inside the quantum paradigm. This task is immediately shown to be not obvious by noticing that emergence and reduction, as concepts involving decomposition into parts, are inherited from a classical way of thinking. Indeed they deeply involve the underlying classical concept of space, a notion which is, strictly speaking, not present in the quantum paradigm.

One of the key idea of the paper will be to propose a definition, strickly inside the quantum paradigm, of decomposition into subsystems which will allow a clear dichotomy between simple and composed states of a given system. This definition is based on physical measurements. Our criterion is, in a sense, a rephrasing inside the quantum paradigm of the natural and classical property of independence between subsystems: a state is decomposable if one can make partial and independent quantum measurements involving statistical features.

This criterion leads to our first main result : being composed by n subsystems is not a property of a system, but a property of its state (exactly as being entangled is not a property of a system, but a property of its state). We also prove, with mathematical rigour, that systems in entangled states are not composed. This is our second main result.

To the question "Are systems in entangled states emergent systems ?" our answer is "No, they are not". Systems in entangled states don't verify our criterion and so are not composed of sub-systems. As a consequence they cannot be considered as examples of emergence, nor as examples of reduction. They are simply not concerned by the debate between emergence and reduction.

The paper is organized as follows: in section 2 we present the notion of emergence and its supposed relevance in quantum mechanics. Section 3 presents the way classical and quantum spaces enters deeply our discussion. In section 4 we describe how entangled states are supposed to be emergent. From the technical definition of decomposability and related results presented in section 5 and linked with their underlying classical equivalent in section 6, we show in 7 that systems that are in entangled states are not examples of emergent systems.

2. Emergence and Quantum Mechanics

2.1. **Emergence in general.** Emergence is a notion which has been introduced as an intermediate notion between reduction and vitalism. The historical debate between reduction and vitalism concerned the living bodies and the following question : can we conceive the living bodies as a pure combination of non-living parts, without anything special? Or: do we have to recognize that living bodies contain something special, something that the non-living bodies don't have? Reductionism chooses the first part of the alternative, vitalism the second one.

This historical debate concerns the relation between a whole and its parts. For reductionists, the behaviour of the whole is deducible from the behaviour of the parts: there is nothing in the whole that is not in the parts. Reductionism is thus the doctrine according to which the behaviour of a compound system is explainable in terms of the behaviour of its parts. In a very approximate manner, we can say that, for reductionists, the whole is nothing more than the sum of its parts. On the contrary, vitalism asserts that the whole is more than the sum of the parts, because living bodies are not only made of physical substance but also of something called "vital principle", "vital force", or sometimes "entelechy", which is absolutely necessary to understand the behaviour of the living body but which is not physical. According to the vitalist, the "vital principle" belongs to the entire body, and not to any of its parts.

Although vitalism is not a priori impossible (it isn't self-contradictory), it suffers from a bad reputation : it has been accused of obscurantism, of being anti-scientific, because the "vital principle" is an hidden principle and doesn't appear directly to our senses (we can only see its effects but not the principle itself). The "vital principle" seems to be an unjustified hypothesis and an obstruction to scientific work.

Is reductionism the unique doctrine that can be accepted to understand the composed systems? The answer is no since the doctrine of emergentism has been constituted. Emergentism conflicts reductionism in the sense that the behaviour of an emergent whole is supposed to be not reducible to the behaviour of the parts. According to emergentism, there is something new in the behaviour of an emergent whole that is not in its parts and that is not a priori deducible from the parts. For example, the behaviour of a living body is not deducible from the behaviour of purely physical matter. But emergentism conflicts also vitalism because it supposes that a living body is entirely made of physical matter and refutes that it has a "vital principle" which would be made of a non-physical substance.

The common intuition that emergentism tries to grasp is that some composed systems present something new that was not in the isolated parts. When these last ones are combined, something new emerges. More precisely emergentism supposes that the behaviour of certain wholes cannot be deduced from the behaviour of its parts taken in isolation and the general law of association. It supposes on the contrary that the behaviour of an emergent whole is new compared to those of its parts and to those of other composed systems that haven't the same parts or that haven't the same structure. For example the behaviour of a living body is supposed to have a kind of autonomy in comparison with the behaviour of its physical parts.

A lot of models of emergence have been proposed by many authors. For a complete review, one can see [8]. In order to clarify these models, it is common to distinguish a strong and a weak form of emergence. The weak form of emergence corresponds to the case where the behaviour of a whole appears to be emergent to us, while in fact it could be reduced to those of the parts. The weak emergence is only due to a waste of knowledge and is often called "epistemological emergence". The strong emergence is called "ontological emergence". A whole is emergent in the ontological sense when its behaviour cannot be reducible (to those of the parts) in principle and not only according to our present knowledge. The strong form of emergence is obviously the most interesting form of emergence and will be the one we'll deal with in this paper.

It is not easy to determine a precise definition of emergence: the common intuition seems not to be sufficient to lead us to a model that everyone can except. We will not discuss it here because we simply need to remark that all the models of emergentism or reductionism concern composed systems and only them: this last point is common to all the models of emergence and reductionism for the simple reason that the debate between these two doctrines concerns the relations between a whole and its parts. If a system doesn't have any parts, that is to say if a system is not composed, there is no sense to wonder whether if this system is emergent or not. That why we won't discuss all these models and won't ask which is the best.

In order to fix the terms of our discussion, we will restrict to the following simple model of emergence (see [8] for more details):

The behaviour of a system can be said emergent if and only if:

- *(a) this system is a composed system, with the structure* $[A_1, ..., A_n; R]$ *(which means : the system is composed of* n *entities* A_i *and these entities are in the relation* R*).*
- *(b) there is a law (called* P_L*) that says that: for all* x*, when* x *has the structure* $[A_1, ..., A_n; R]$ *then* x *has the behaviour* C*.*
- *(c)*P_L *cannot be deduced from the laws concerning the isolated entities* $A_1, ..., A_n$*, nor from the laws concerning combined systems that contain some of (but not all) these entities* $A_1, ..., A_n$*.*

Out of this model it is easy to build a model of reduction: we only have to change point (c) in order that P_L can be deduced from the laws concerning the isolated entities $A_1, ..., A_n$, or from the laws concerning composed systems that contains some of (but not all) these entities $A_1, ..., A_n$.

Let us say it again: in this article we don't want to discuss points (b) and (c). We only need to remark that point (a) is necessary to all the models of emergence or of reduction.

2.2. **Link with Quantum Mechanics.** Why is quantum mechanics involved in this debate? Our answer is that quantum mechanics seems to solve the two kinds of difficulties that emergentism meets. The first kind of problems only concerns the purely conceptual aspect of emergentism. This part of the discussion can be summed up in this question: "is emergentism, when correctly defined, an acceptable metaphysical doctrine?" Or: "is it a priori possible for a system to be emergent and in what sense?" We'll call this question "Q1". The principal criticisms to emergentism came from Jaegwon Kim (see in particular:[3]) and gave rise to a big literature.

The second kind of problems is less general and concerns the existing systems. The question (called "Q2") is the following: does it really exist at least one system that is emergent? This question can then be asked in the different domains of scientific investigation: biology (do the living bodies emerge from their physical basis?), philosophy of mind (do the mental properties emerge from the physical properties?), sociology (does a social fact emerge from individual facts?), economy (does the behaviour of the market emerge from the behaviour of an isolated economical agent?), etc.

The hope of the advocates of emergentism is that Quantum Mechanics allows us to demonstrate rigorously that some quantum systems are emergent. The stake in this point is big: if it is possible to find in Quantum Mechanics some systems that are emergent and if it is possible to demonstrate this result, then we will answer the two questions Q1 and Q2 by 'yes'. It would allow us to prove that emergentism is not only an acceptable metaphysical doctrine (Q1), but also that it describes some part of our experience (Q2). The mathematical formalism of Quantum Mechanics and the particular property of entanglement would justify emergentism in all its aspects and in a way that would not be questionable. If we can prove that systems in entangled states are emergent, then we prove emergence! This is why the advocates of emergentism paid so much attention on Quantum Mechanics.

3. A crash course in Quantum Mechanics

One point will be important for our demonstration and must be emphasized: it is the role of space-state in measurement. This point is rarely mentioned while it can cause confusion. A measurement in Quantum Mechanics, as in Physics in general, supposes the interaction between two physical systems : the systems S that we want to measure and the measurement apparatus. We must be able to describe this interaction in the physical space. The physical space represents some new degrees of freedom of any system and as every degree of freedom, we have to use an Hilbert space to describe them. If we want to describe the physical space in a three-dimension space, we'll use the Hilbert space of the square integrable functions (that is noted $L^2(\mathbb{R}^3)$).

Let us suppose for example that a system S is a $\frac{1}{2}$-spin particle. We need two Hilbert spaces: one for the spin and one for the space-state. It comes from it that the global Hilbert space will be: $H = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$. Thus the general form of the state vector will be any vector of H. For example if the particle is localized in a region of the physical space and is in an entangled state for the values of spin, the state vector will be of this form: $|\Psi\rangle = \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle] \otimes \alpha(X)$, where $\alpha(X)$ is a function of $\mathbb{R}^3$ that is equal to zero everywhere excepted in the region where the particle is localized.

If we want now to describe the operators that can be used for a measurement, we have to specify also the space-state of the operator. In other terms the operators that we have to use must be defined in the Hilbert space H. For example if σ is an operator of $\mathbb{C}^2$, the operator that we have to write is: $\Sigma = \sigma_1 \otimes \beta(X)$, where $\beta(X)$ describes the space-state of the apparatus.

It is now simple to describe basic situations.

- When the apparatus is off, $\beta(X) = 0$.

- If the supports of the two functions $\alpha(X)$ and $\beta(X)$ are disjoint, no measurement can be done. It simply means that there isn't any interaction between S and the apparatus.
- If $\beta(X) = 1$ in the region when $\alpha(X)$ isn't equal to zero, we have $\alpha(X).\beta(X) = \alpha(X)$. Then:

$$\Sigma|\Psi\rangle = [\sigma.(\frac{1}{\sqrt{2}}[|+\rangle + |-\rangle]] \otimes \alpha(X).$$

In this expression the term $\sigma.(\frac{1}{\sqrt{2}}[|+\rangle + |-\rangle$ represents the action of the operator on the system S when we don't mention the space-state of S and of the apparatus.

How can we now describe the situation with two identical systems ? Let us suppose for example that we want to describe the system S composed by two particles of spin $\frac{1}{2}$ noted S_1 and S_2 . Let us suppose also that each of them is represented by a state vector $|\psi_1\rangle$ and $|\psi_2\rangle$. The global Hilbert space that describes the spin-state of the composed system S is $\mathbb{C}^2 \otimes \mathbb{C}^2$. If the two particles are initially independent from each other, the state vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$ will be $|\psi_1\rangle \otimes |\psi_2\rangle$. But we have to describe also the space-state of the particles. The Hilbert space will thus be: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. If $f_1(X)$ and $f_2(X)$ are two functions that describe the space-state of (respectively) S_1 and S_2, the state vector of S will be:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes f_1(X) \otimes f_2(X)$$

To work in a situation a little bit more general, we'll suppose now that the spin-state of S can be in an entangled state. For reason of simplicity we'll also suppose that the two space-functions f_1 and f_2 are not entangled. The state vector of S will thus be:

$$|\psi\rangle = |u\rangle \otimes f_1(X) \otimes f_2(X)$$

Let us suppose that we want to measure the spin of the particles with a Stern-Gerlach device. In the Hilbert space H the corresponding operator will be:

$$\Sigma = \sigma \otimes Id_{\mathbb{C}^2} \otimes \chi(X) \otimes Id_{\mathbb{R}^3} + Id_{\mathbb{C}^2} \otimes \sigma \otimes Id_{\mathbb{R}^3} \otimes \chi(X)$$

where

- σ is the operator to the Stern-Gerlach device in the Hilbert space $\mathbb{C}^2$.
- $\chi(X)$ is the function that represent the space-state of the Stern-Gerlach device in $L^2(\mathbb{R}^3)$

If $(f_1.\chi)(X) \neq 0$ and $(f_2.\chi)(X) \neq 0$, the Stern-Gerlach device will have an effect on the two particles and will measure their spin. More precisely the effect of the operator Σ on the state vector $|\Psi\rangle$ is :

$$\Sigma|\Psi\rangle = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes [\chi.f_1(X)] \otimes [I_{\mathbb{R}^3}.f_2(X)] + [(I_{\mathbb{C}^2} \otimes \sigma)|u\rangle] \otimes [I_{\mathbb{R}^3}.f_1(X)] \otimes [\chi.f_2(X)]$$

Is it possible to measure only one particle? The only possibility consists in having the second term equal to zero. In order to do that, we have to suppose that the support of the two spatial functions $f_2(X)$ and $\chi(X)$ are disjoint, and by consequence that $f_2(X).\chi(X) = 0$. If we suppose also that the spatial function $\chi(X)$ is equal to 1 everywhere the function $f_1(X)$ is not equal to zero, and by consequence that $f_1(X).\chi(X) = f_1(X)$, we have :

- $[(I_{\mathbb{C}^2} \otimes \sigma)|u\rangle] \otimes [I_{\mathbb{R}^3}.f_1(X)] \otimes [\chi.f_2(X)] = 0$
- $[(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes [\chi.f_1(X)] \otimes [I_{\mathbb{R}^3}.f_2(X)] = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes f_1(X) \otimes f_2(X)$

And finally :

$$\Sigma|\Psi\rangle = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes f_1(X) \otimes f_2(X)$$

If we want to measure the spin of a particle that is entangled with an identical one, this is the only way to do it.

4. Entanglement considered as a case of emergence

A lot of authors have argued that systems in entangled states are emergent systems and their argumentations are always based on the same ideas. We can call this kind of argumentation 'the traditional argumentation' because this argumentation has now become very common and broadly accepted. This traditional argumentation compares the situation of a factorized vector and an entangled vector. Let us suppose that we have two distinct particles independent from each other and that they can be represented by two state vectors $|\phi\rangle$ and $|\Phi\rangle$ which belong to two Hilbert spaces (resp.) H_1 et H_2. According to Quantum Mechanics the global system S lives in a Hilbert space $H = H_1 \otimes H_2$ and its state vector is $|\Sigma\rangle = |\phi\rangle \otimes |\Phi\rangle$. This vector is a tensor product and so it has a factorized form. According to the traditional argumentation we have here a simple example of resultant system: the state of the global system S is reducible to the state of its two parts because its state vector can be written as a product of two state vectors.

What happens now if system S is in an entangled state? For example let us suppose that the two particles have come closer and have interacted in such a way that the state is now entangled. Its state vector $|\Sigma\rangle$ cannot be factorized anymore and it is now a superposition of factorized vectors. For example it can be of this form: $|\Sigma\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle)$. Here it is impossible to give any state to the (supposed) parts of system S, that is to say: it is impossible to give any state to the particles that produced system S. So it is also impossible to reduce the state of the whole to the states of its (supposed) parts. According to the traditional argumentation this is a case of emergence. For example Andreas Hüttemann explains in [1] that:

> Quantum entanglement is a counterexample to synchronic micro-explanation [...]. The compound [system consisting of two non-identical particles] is in a determinate state, but this cannot be explained in terms of the determinate states of its constituents. This is because there are states [...] that do not allow the attribution of pure states to the parts of the compound. What we see is that synchronic micro-explanation systematically fails. Thus, we have an example of emergence [...]. This is a case of emergence because it is, in principle, impossible to explain the behaviour of the compound (in this case: the state) in terms of the behaviour (states) of the parts.

A lot of authors has supported this argumentation. As we said in introduction, one can also refer to [9], [7], [2], [5],[6], [4].

5. A Notion of decomposition into subsystems inside the quantum paradigm

As we said emergence and reduction are notions that concern composed systems: they are supposed to tell us what kind of relations exist between a whole and its parts. Thus there is no sense to wonder whether a system is emergent or not if we cannot identify any parts inside this system, that is to say if this system is simple and not composed. A system that is not composed is neither emergent nor resultant: it just can not enter this discussion.

But what is less obvious is to determine if the entanglement systems are really composed systems or if they are actually simple systems. The traditional argumentation assumes that systems in entangled states are composed since it concludes that they are emergent but usually it isn't said explicitly why we have to consider them as composed systems.

So we need a criterion to clearly distinguish between simple systems and composed systems. But we would like a criterion that would be independent (or as independent as it is possible) from metaphysical assumptions or ontological choices. For example with a corpuscular ontology a unique corpuscle is a simple system, and a composed system contains at least two corpuscles. The ontology fixes the criterion of composed system. But what we are looking for in this paper is a criterion that could potentially be accepted by everyone and that avoids metaphysical choices (or too heavy metaphysical choices). To achieve this aim, our solution consist in basing our criterion on the practice of the physicist. If we can find the justifications of our criterion inside the pratice of the physicist, then our criterion will not have been chosen because of some metaphysical reasons or general principles choosen outside physics. Of course this is also a philosophical choice.This is a pragmatist way of thinking: the philosophical principles or concepts are based on practice and a lot of attention is devoted to the practical consequence of the philosophical thesis.

What can such a criterion be?

The first part of our answer will be that a system is composed provided there exists acts of knowledge that can be decomposed part by part. By act of knowledge we mean any measurement that can be done on it. By decomposable act of knowledge we mean any measurement than can be carried out on any part of the system and assigns a given value to each of them. In other words, it must be possible to measure one part and not the others, that it to say it is possible to make partial measurements, therefore to distinguish different parts or sub-systems which correspond to the different partial measurements that can be done. If the system cannot be measured part by part, there is no practical interest in considering this system as a composed system: the property of being composed is then purely theoretical. To resume, this first condition can be expressed as follow : it must be possible to make partial measurements on the system, one partial measurement for each part of the system. At this stage it is natural to say that a non-emergent system is such that any act of knowledge is somehow decomposable.

But this condition is not the only one. Indeed let us suppose that we have a system S composed of two parts $S1$ and $S2$ and that we make a first partial measurement ($M1$) on the part S1 and then a second partial measurement ($M2$) on the other part $S2$. Let us suppose now that the partial measurements are not independent from each others, that is to say that the results of $M2$ depend on the result of a earlier partial measurement $M1$. Then it means that the behavior of one part is intrinsically dependent on the behavior of some other parts. For the practice of the physicist there is no interest in considering that, in this situation, the system S would truly be composed: it would be logically possible but it wouldn't have any impact on the practice. In other words it seems to us that there is no practical reasons for which we would consider such a system as composed. That's why we put a second condition for our criterion: the measurements have to provide independent results. The fact that we made a first measurement on one part and obtained a given result must have no consequence on the results by any further partial measurement.

So we can give a first formulation of our criterion (a more accurate formulation will be given later):

A system S is composed of n parts S_i if and only if

- *Condition 1: for any part S_i it is possible to realize a partial measurement M_i that concerns only S_i;*
- *Condition 2: all the partial measurements are independent from each other, that is to say that the results of any partial measurement M_i don't depend on the fact that other partial measurement M_j (with $j \neq i$) would have eventually been made and don't depend on the results of M_j.*

Are the two conditions sufficient? We think that they are sufficient because systems that verify these two conditions allow the physicist to decompose the acts of knowledge that we can operate on it. Physicists can study each part of the system as we want, independently from what has been done before (independently from any partial measurement that would have been made before).

It must be also emphasized that at this point our criterion doesn't depend on Quantum Mechanics. It doesn't use any concept that belongs to Quantum Mechanics, it isn't formulated inside the quantum paradigm, and it isn't justified by some reasons belonging to the Quantum Mechanics. That means that our criterion is more general that Quantum Mechanics and can be used in other contexts.

We will discuss the relevance of this criterion in the classical framework in Section . But since we want to use this criterion inside the quantum paradigm we need to give a formulation inside this paradigm.

5.1. **The criterion.** In order to simplify the formulation, we will restrict ourselves to the 2×2 dimension. That means that we will focus on systems composed of only two parts which can be described by 2-dimensional Hilbert spaces.

Let σ_1 (resp. σ_2) be an hermitian operator acting on the 2-dimension Hilbert space H_1 (resp. H_2). To σ_1 (resp. σ_2) we associate the operator Σ_1

(resp Σ_2) on $H = H_1 \otimes H_2$ defined by

$$\Sigma_1 = \sigma_1 \otimes I_{H_2} \text{ (resp. } \Sigma_2 = I_{H_1} \otimes \sigma_2) \tag{1}$$

Definition 5.1. We define:

- a Σ_1-measurement (resp. a Σ_2-measurement) is a measurement that corresponds to the hermitian operator Σ_1 (resp. Σ_2)
- $P(\Sigma_1 = \lambda)$ (resp. $P(\Sigma_2 = \lambda'))$ is the probability that the result of the Σ_1-measurement (resp. the Σ_1-measurement) is λ (resp. λ').
- $P_{\Sigma_2=\lambda'}(\Sigma_1 = \lambda)$ (resp. $P_{\Sigma_1=\lambda}(\Sigma_2 = \lambda^{'})$) the probability that the result of the Σ_1-measurement (resp. the Σ_2-measurement) if λ (resp. λ') after a Σ_2-measurement (resp. a Σ_1-measurement) has been performed and given the result λ' (resp. λ).

We can now state our

Criterion of decomposition

The system S that is described by a state vector in H is a system composed of the 2 sub-systems S_1 and S_2 which are supposed to live respectively in the Hilbert spaces H_1 and H_2 if and only if:

- *Condition 1: it is possible to make a Σ_1-measurement of S and a Σ_2-measurement of S.*
- *Condition 2:* $\forall(\Sigma_1, \Sigma_2)$, $\forall(\lambda, \lambda^{'})$
$$P(\Sigma_1 = \lambda) = P_{\Sigma_2=\lambda'}(\Sigma_1 = \lambda)$$
$$P(\Sigma_2 = \lambda^{'}) = P_{\Sigma_1=\lambda_1}(\Sigma_2 = \lambda^{'})$$

Let us now give an explanation of the technical aspects of this criterion. The Condition 1 concerns the physical possibility of partial measurements. What we said is that a composed system can be studied part by part, that is to say that it is possible to realize physically a separated measurement, a measurement that concerns only $S_i (i = 1 \dots n)$ and not the other parts $S_j (j \neq i)$ of S. The problem here is that the state vector of the system S belongs to H : any measurement that we could make must be described in the same Hilbert space. The solution is to use an operator defined in H but that will have effect only on the sub-space H_i. The general form of this kind of operator is: $\Sigma_i = I_1 \otimes I_2 \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes I_n$ (where σ_i is a hermitian operator in H_i). This kind of operator concerns only H_i and has no effect on any other sub-system H_j (for $j \neq i$ they correspond to the identity-operator).

The object of Condition 2 is the results of the separated measurements. As we say those results must be independent from each other. In Quantum Mechanics the prediction can only be probabilistic. So the independence of the results must concern the probabilities of results for any separated measurements: the function of probabilities of results for any partial measurement must be the same whether or not an other partial measurement has been done before. The independence of results of the measurements is so guaranteed by the independence of the probabilities of the measurements. That's why Condition 2 asserts that the probabilities of results for any measurement Σ_i must be the same whether or not a Σ_j-measurement ($j \neq i$) has been done.

Now the question is: are systems in entangled states composed systems? To answer this question we only need to see if systems in entangled states verify the two conditions.

5.2. **The mathematical results.** The result of our work consists in two mathematical results whose demonstrations are given in appendices A and B respectively.

The first one concerns the Condition 1: **Result 1.**

In the situations where $H_1 = H_2$, the partial measurements are not possible if the system doesn't have a space function that is a tensor product of two disjoint space functions. Let us suppose for example that two identical systems interacts and that the state of the global system becomes entangled. Then if the system is spatially localized in only one region, it is impossible to make partial measurements. Such a system is not a composed system.

We think that this result is interesting because it shows that quantum paradigm is able to describe how it is possible to make partial measurements. If we want to use our criterion inside Quantum Mechanics we need to formulate everything inside its paradigm, so it is important to describe exactly the mathematical and physical conditions to verify Condition 1. Furthermore, this condition about the space function of the system is often forgiven while it is necessary to make the partial measurements.

The second mathematical result is the following:

Theorem 5.2. *We have*

$$\begin{gathered} |\Psi\rangle \textit{ satisfies Condition 2} \\ \Updownarrow \\ |\Psi\rangle = |u\rangle \otimes |v\rangle \textit{ with } |u\rangle \in H_1 \textit{ and } |v\rangle \in H_2. \end{gathered}$$

It is obvious that all the tensor product of two vectors of the Hilbert spaces H_1 et H_2 verifies Condition 2. But the reverse relation is less obvious. For details one can see the demonstration in appendix B.

This relation means that a system S that is in an entangled state cannot satisfy condition 2. Thus according to our criterion such a system is not a composed system. We'll comment this result in the two next sections.

5.3. **Generalization.** Let us give in this section the generalization of our previous results in the cases of multicomponents decompositions and Hilbert spaces of higher dimensions.

The criterion becomes the following.

Let σ_i be hermitian operators acting on the Hilbert spaces H_i. To σ_i we associate the operator Σ_i on $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ defined by

$$\Sigma_i = I_{H_1} \otimes I_{H_2} \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes I_{H_n} \tag{2}$$

Definition 5.3. We define:

- a Σ_i-measurement is a measurement that corresponds to the hermitian operator Σ_i
- $P(\Sigma_i = \lambda)$ is the probability that the result of the Σ_i-measurement is λ
- $P_{\Sigma_j=\lambda'}(\Sigma_i = \lambda)$ the probability that the result of the Σ_i-measurement is λ after a Σ_j-measurement has been performed and has given the result λ'.

We can now state our

Criterion of decomposition

The system S that is described by a state vector in H is a system composed of the n sub-systems $S_1, S_2, \ldots, S_n$ which are supposed to live respectively in the Hilbert spaces $H_1, H_2, \ldots H_n$ if and only if:

- *Condition 1: it is possible to make Σ_i-measurements of S.*
- *Condition 2: $\forall(\Sigma_i, \Sigma_j)(j \neq i), \forall(\lambda, \lambda')$, we have:*

$$P(\Sigma_i = \lambda) = P_{\Sigma_j=\lambda'}(\Sigma_i = \lambda)$$

The mathematical results follow the same way in this situation, see Appendix C and D for the proofs.

6. Link with "classical" partition

In this section we will emphasize some important differences between the situation in Quantum Mechanics according to our criterion and the situation in classical physics or in daily life.

The criterion we propose here is based on the idea that a system is composed if and only if it can be studied part by part, by partial measurements that must be independent. This idea corresponds to the intuition that a clock, by example, is a composed system because we can observe and study the behaviour of different parts: a spring, the hour hand, the minute hand, a wheel, etc. The behaviour of the clock can be divided into many parts that can be studied independently. We don't want to say that the behaviour itself of each part is independent (which would be wrong). We just want to say that it is possible to study part by part and begin with any part of the clock without any consequence on the measurements we'll make on the other parts. So the definition of our criterion seems to correspond to our common intuition about what a composed system is.

But its consequences are rather different from what is commonly thought. First space doesn't play its classical role anymore. In classical Physics, physical matter is conceived in a strong link with space in the sense that two different pieces of matter are supposed to occupy two different regions of space. Physical matter is thus said to be impenetrable. This property has a long story in the history of philosophy: Aristotle, the Stoicians, the epicureans, but also Galileo and Newton claimed that being impenetrable is one the properties of matter. According to all of them, two systems must be in two regions of space. Furthermore, this property can lead us to think that, on the contrary, if a system occupies two separated regions of space, then it must be considered as a system composed from two different systems. In other word, a simple system is either here or there, but not in the two places at the same time. Space can then play an important role to count the number of systems. Each separated region of spaces can be associated with one system. Then the number of such regions of space is also the number of parts that the system contains. According to us, this is the first reason why systems in entangled states are usually considered as composed systems. But this reason is not good because the role that space plays in Quantum Mechanics is very different from the one it plays in classical Physics. In Quantum Mechanics, space is treated as the other degrees of freedom, that is to say as spin or as polarization. That's why the space function of a system can be entangled with the spin or can be distributed in two separated continuous functions. In Quantum Mechanics it is wrong to think that a system can be associated with only one region of space and that one region of space can be associated with only one system. Space doesn't have the specific status it has in classical Physics.

The second reason why systems in entangled states are usually considered as composed systems is that it can be produced by two systems that are originally independent. For example we can take two different and independent electrons and then create entanglement by making them interact. Then we are tempted to consider that after interaction there are still two systems because they were two systems before. This is right for usual objects: the engine and the wheels of a car can interact, they are still different systems.

In our point of view this is no more the case in Quantum Mechanics. It is a consequence of our criterion that two systems can become only one system. Any interaction that leads to entanglement is like a fusion of the two systems that don't exist anymore during entanglement. This situation correspond to what can happen in biology: for example two cells can fusion and leads to a new cell. In sociology also two systems (for example two social groups) can fusion into one single system (one new social group) and disappear. One other aspect must be emphasized in Quantum Mechanics: a measurement on an entanglement system leads necessarily to a state vector that can be factorized. In other words, because of the measurement the parts that initially composed the system and that disappeared can now exist again. This is also the case in sociology for example because two groups can disappear and then be formed again.

Let us call the 'composability' of a system the fact that it is or not a composed system. Now we can compare the composability in classical domain and in Quantum Mechanics. One feature of the classical composability is that it doesn't depend on time. If a system is a composed system, it will remain a composed system. The classical composability is then a property of the system (or of its nature, which cannot change) and not of this state (which depends on time). In Quantum Mechanics the composability can change in time: for example a system can be initially in a factorized state and then become entangled. The quantum composability is then a characteristics of the state and not of the nature of the system like for classical system.

But the link between quantum composability and classical composability is not like the link between quantum quantity and classical quantity (for example between the position in Quantum Mechanics and the position in classical Physics). In classical physics the state of a system is described by physical quantities as its mass, its volume, etc. These physical quantities are the properties of the system. Each property corresponds to one value of a physical quantity. So the state of a classical system is determined by the values of all its properties. To be a simple system or a composed system is also a property: a system is either a simple system or a composed system. Let us call this property the 'classical composability'.

In Quantum Mechanics the physical quantities are the observable. In general if we measure one observable, we can obtain many results because the system can be in a state that corresponds to a superposition of the different values we can obtain. In other words, a 'quantum property' is characterized by the plurality of values at the same time and by the principle of superposition. Is quantum composability a quantum quantity? Like the spin and other quantum quantities, the quantum composability can take two values for the same system because its state can change in time. But the two values that can take the quantum composability cannot be superposed. Then the answer is no: quantum composability is not a quantum quantity like the spin or the polarization.

7. Back to emergence and reduction

The conclusion of our analysis is that a system in an entangled state is not an emergent system. Any system in entangled state doesn't verify the first hypothesis of any model of emergence, that is to say that the system is a composed system (this hypothesis corresponds to point (a) of our model of emergentism). Thus emergentism cannot argue that systems in entangled states are examples of emergent systems, or that Quantum Mechanics provides a demonstration that emergent systems exist.

But we don't want to say that the behaviour of systems in entangled states could be explained by the behaviour of their parts and thus could be reduced to the latest: we say that systems in entangled states don't have any parts. In other terms this means that the debate between emergentism and reductionism doesn't concern systems in entangled states. So our conclusion cannot be used in favor of reductionism and against emergentism: it cannot be used by any of the two doctrines.

We would like to insist on that point: this study doesn't take any position against emergence in general. The fact that systems in entangled states are not emergent systems doesn't mean that there is no emergent system in nature. Our point is just that systems in entangled states do not fit the conceptual frame of this debate. In others terms: the criterion we propose here does not lead to any answer to Q1 and Q2 (see 2.2).

We would like to make one remark to conclude. Our criterion leads to the result that a system that is produced by two initially independent subsystems can become a non-composed system. As we said in the previous section, it is as if these two subsystems have fused and disappeared as systems: they don't exist anymore. This result could remind us of the model proposed by Paul Humphreys in [2]. In this model, emergence is conceived as a process of fusion. At the beginning they are some instances of property of a first level (called the i-level). But at the end, they fusion and give birth to instances of properties of a higher level (the i+1-level). The instances of properties of the i-level don't exist anymore: the instances of properties of the i+1-level are the only ones to exist at that time. For that reason it has become impossible to reduce the instances of the properties of i+1-level to the instances of properties of i-level. As he says,

> when emergence occurs, the lower level property instances go out of existence by producing the higher level emergent instances

In that sense this operation of fusion can be viewed as a case of emergence. Then the author gives the example of systems in entangled states. For him, this kind of systems corresponds to the model of fusion he described: systems in entangled states are emergent systems according to his model of emergence by fusion. As he says:

> The composite system can be in a pure state when the component systems are not, and the state of one component cannot be completely specified without reference to the state of the other component. Furthermore, the state of the compound system determines the states of the constituents, but not vice versa [...]. I believe that the interactions which give rise to these entangled states lend themselves to the fusion treatment described in the earlier part of this paper, because the essentially relational interactions between the 'constituents' (which can no longer be separately individuated within the entangled pair) have exactly the features required for fusion.

In the model of fusion proposed by Humphreys, the instances of properties fusion but not the entities that support these properties. The entities of i-level still exist even if the properties of i-level don't exist anymore. This aspect of his model is not surprising because there is no sense to talk about emergence if they are not parts of the global system or if this system doesn't have parts anymore. Humphreys explicitly says that it is possible that an object of i+1-level occurs in the fusion operation but it is not necessary. But even in this case he doesn't say that the i-level entities disappear. That's why in the preceding citation he talks about the component of a system in entangled state and supposes then that they still exist. He just remarks that the component of an entangled pair cannot be separately individuated. But this remark doesn't lead him to the conclusion that they don't exist anymore (and then that there is no emergence at all). So the difference between Humphreys and us is that he supposes that the parts still exist whereas we think that they don't exist anymore. That why we claim that systems in entangled states are not emergent systems.

Appendix A. Proof of Result 1

How is it possible in Quantum Mechanics to physically realize the operator $\Sigma_i = I_1 \otimes I_2 \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes I_n$? For example let's suppose that the system S is produced by two particles of spin $\frac{1}{2}$. The Hilbert space that describes the spin-spaces is $\mathbb{C}^2 \otimes \mathbb{C}^2$. Let's suppose now that we want to measure the spin of only one particle with a Stern-Gerlach device. The problem is that in this situation there is no reason why the Stern-Gerlach device measures only one particle and not the other. More precisely this kind of measurement will be formalized by this operator:

$$\Sigma = \sigma_1 \otimes I_2 + I_1 \otimes \sigma_2$$

Here the operator will act on the two particles.

How is it possible to transform $\Sigma = \sigma_1 \otimes I_2 + I_1 \otimes \sigma_2$ in $\Sigma = \sigma_1 \otimes Id_2$ which is the kind of operator we want to realize? The way to do this consists in using spatial functions: we'll describe not only the spin state of the particles but also their space-state. In order to do that we must introduce the Hilbert sub-space $L^2(\mathbb{R}^3)$ which is the space of the square integrable functions. Suppose we can write $H = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. It is then possible to realize the kind of operator we want to do. Let's write $\chi(X)$ the function describing in $L^2(\mathbb{R}^3)$ the space-state of the measuring device (for example the space-state of the Stern-Gerlach device). If we worked with only one particle of spin $\frac{1}{2}$ the Hilbert space would be $H = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ and the hermitian operator of the Stern-Gerlach device would be $\alpha = \sigma \otimes \chi$.

But we work in the global Hilbert space $H = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. So the hermitian operator will be :

$$\alpha = \sigma \otimes I_{\mathbb{C}^2} \otimes \chi \otimes I_{\mathbb{R}^3} + I_{\mathbb{C}^2} \otimes \sigma \otimes I_{\mathbb{R}^3} \otimes \chi$$

Suppose now that the state vector of the system S is :

$$|\Psi\rangle = |u\rangle \otimes f_1(X) \otimes f_2(X),$$

where $|\Psi\rangle$ corresponds to the spin-state of S ($|\Psi\rangle$ belongs to the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$) and $f_1(X) \otimes f_2(X)$ correspond to the space-states of the particles ($f_1(X) \otimes f_2(X)$ belongs to the Hilbert space $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$)

The effect of the operator α on the state vector $|\Psi\rangle$ is :

$$\alpha|\Psi\rangle = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes [\chi.f_1(X)] \otimes [I_{\mathbb{R}^3}.f_2(X)] + [(I_{\mathbb{C}^2} \otimes \sigma)|u\rangle] \otimes [I_{\mathbb{R}^3}.f_1(X)] \otimes [\chi.f_2(X)]$$

The problem will be solved if the second term is equal to zero. The way to do that consists in supposing that

- the support of the two spatial functions $f_2(X)$ and $\chi(X)$ are disjoint, and by consequence that $f_2(X).\chi(X) = 0$;
- the spatial function $\chi(X)$ is equal to 1 everywhere the function $f_1(X)$ is not equal to zero, and by consequence that $f_1(X).\chi(X) = f_1(X)$.

With this assumption we have :

- $[(I_{\mathbb{C}^2} \otimes \sigma)|u\rangle] \otimes [I_{\mathbb{R}^3}.f_1(X)] \otimes [\chi.f_2(X)] = 0$
- $[(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes [\chi.f_1(X)] \otimes [I_{\mathbb{R}^3}.f_2(X)] = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes f_1(X) \otimes f_2(X)$

And finally :

$$\alpha|\Psi\rangle = [(\sigma \otimes I_{\mathbb{C}^2})|u\rangle] \otimes f_1(X) \otimes f_2(X)$$

Now it is very simple to find the eigenvectors of the operator α . In general, if $\Sigma = \sigma \otimes \rho$ then $Spect(\Sigma) = Spect(\sigma).Spect(\rho)$ and the eigenvectors of Σ are all the tensor products of a eigenvector of σ with an eigenvector of ρ.

So here we have $Spect(\sigma.I_{\mathbb{C}^2}) = Spect(\sigma)$ and the eigenvectors of $\sigma.I_{\mathbb{C}^2}$ are all the vectors of the form: $|u_\lambda\rangle \otimes |v\rangle$ (it is easy to see that : if $|u_\lambda\rangle$ is an eigenvector of σ associated with the eigenvalue λ, and if $|v\rangle$ is a vector of the Hilbert space $\mathbb{C}^2$, then: $[\sigma \otimes I_{\mathbb{C}^2}][|u_\lambda\rangle \otimes |v\rangle] = [\sigma \otimes |u_\lambda\rangle] \otimes |v\rangle = \lambda |u_\lambda\rangle \otimes |v\rangle$).

If now we note $|U_\lambda\rangle_{f_1 f_2}$ the vectors of this form:

$$|U_\lambda\rangle_{f_1 f_2} = |u_\lambda\rangle \otimes |v\rangle \otimes f_1(X) \otimes f_2(X),$$

we have :

$$\alpha|U_\lambda\rangle_{f_1 f_2} = [\sigma \otimes I_{\mathbb{C}^2} \otimes \chi \otimes I_{\mathbb{R}^3}].[|u_\lambda\rangle \otimes |v\rangle \otimes f_1(X) \otimes f_2(X)] = [\sigma \otimes |u_\lambda\rangle] \otimes f_1(X) \otimes f_2(X)],$$

And finally:

$$\alpha|U_\lambda\rangle_{f_1 f_2} = \lambda |U_\lambda\rangle_{f_1 f_2},$$

which means that all the vectors $|U_\lambda\rangle_{f_1 f_2}$ are eigenvectors of α.

Appendix B. Proof of Theorem 5.2

B.1. **Main result.** As we said, we'll work in a 4-dimension Hilbert space H. Let us take a system S that is described by a state vector $|\Psi\rangle$ and suppose that we want to analyze it in two sub-systems S_1 et S_2 that live in two 2-dimensional Hilbert spaces H_1 and H_2 (with $H = H_1 \otimes H_2$). We won't suppose the system S is in an entanglement state. On the contrary we'll show that :

The state vector $|\Psi\rangle$ of a system S verifies Condition 2 $\Leftrightarrow$ $|\Psi\rangle$ is a tensor product of two vectors of the Hilbert spaces H_1 and H_2.

Let's suppose that σ_1 and σ_2 are two hermitian operators belonging (resp.) to H_1 and H_2. We'll note $|\pm\rangle_1$ and $|\pm\rangle_2$ their eigenvectors. Each pair of eigenvectors is an orthonormal basis. So we can write :

$$|\Psi\rangle = c_{++}|++\rangle + c_{+-}|+-\rangle + c_{-+}|-+\rangle + c_{--}|--\rangle \tag{3}$$

The condition for $|\Psi\rangle$ to be a tensor product is:

$$c_{++}c_{--} = c_{+-}c_{-+} \tag{4}$$

We suppose that $|\Psi\rangle$ verifies Condition 2. The general relation $P(\Sigma_i) = P_{\Sigma_j}(\Sigma_i)$ with $(j \neq i)$ must be right for any operator Σ_i and Σ_j. So it must be right in particular for the two operators σ_i and σ_j, and we can write $P(\Sigma_2) = P_{\Sigma_1}(\Sigma_2)$, with $\Sigma_1 = \sigma_1 \otimes I_2$ and $\Sigma_2 = I_1 \otimes \sigma_2$

From the expression 30 we have

$$P(\Sigma_2 = +) = |c_{++}|^2 + |c_{-+}|^2$$
$$P(\Sigma_2 = -) = |c_{+-}|^2 + |c_{--}|^2$$

What are the probabilities of a Σ_2-measurement if we made a Σ_1-measurement before it? The result of the Σ_1-measurement can be either "+" or "-".

- If the result is "+", the state vector is:

$$|\Psi\rangle = \frac{1}{\sqrt{|c_{++}|^2 + |c_{+-}|^2}}|+\rangle \otimes [c_{++}|+\rangle + c_{+-}|-\rangle]$$

- Otherwise the state vector is:

$$|\Psi\rangle = \frac{1}{\sqrt{|c_{-+}|^2 + |c_{--}|^2}}|-\rangle \otimes [c_{-+}|+\rangle + c_{--}|-\rangle]$$

Then we make the Σ_2-measurement. We have:

$$P_{\Sigma_1=+}(\Sigma_2=+)=\frac{|c_{++}|^2}{|c_{++}|^2+|c_{+-}|^2}$$

$$P_{\Sigma_1=-}(\Sigma_2=+)=\frac{|c_{-+}|^2}{|c_{-+}|^2+|c_{--}|^2}$$

According to the Condition 2, we must have:
$P(\Sigma_2=+)=P_{\Sigma_1=+}(\Sigma_2=+)=P_{\Sigma_1=-}(\Sigma_2=+)$, that is to say:

$$\frac{|c_{++}|^2}{|c_{++}|^2+|c_{+-}|^2}=\frac{|c_{-+}|^2}{|c_{-+}|^2+|c_{--}|^2}=|c_{++}|^2+|c_{-+}|^2$$

and $P(\Sigma_2=-)=P_{\Sigma_1=+}(\Sigma_2=-)=P_{\Sigma_1=-}(\Sigma_2=-)$,

$$\frac{|c_{+-}|^2}{|c_{++}|^2+|c_{+-}|^2}=\frac{|c_{--}|^2}{|c_{-+}|^2+|c_{--}|^2}=|c_{+-}|^2+|c_{--}|^2$$

From these relations we have

(5) $$|c_{++}|^2.|c_{--}|^2=|c_{+-}|^2.|c_{-+}|^2$$

Here we have supposed that Condition 2 is true. Let us remind that Condition 2 is :

(6)
$$\forall(\Sigma_1,\Sigma_2),\forall(\lambda_1,\lambda_2^{'}),P(\Sigma_1=\lambda)=P_{\Sigma_2=\lambda^{'}}(\Sigma_1=\lambda),P(\Sigma_2=\lambda^{'})=P_{\Sigma_1=\lambda_1}(\Sigma_2=\lambda^{'})$$

But up to this point we haven't used all the implications of Condition 2. In fact we choose one couple of operators Σ_1 and Σ_2 that verify the relation

(7) $$\forall(\lambda_1,\lambda_2^{'}),P(\Sigma_1=\lambda)=P_{\Sigma_2=\lambda^{'}}(\Sigma_1=\lambda),P(\Sigma_2=\lambda^{'})=P_{\Sigma_1=\lambda_1}(\Sigma_2=\lambda^{'})$$

But if Condition 2 is true, the relation 35 is supposed to be true with every hermitian operators σ_1 and σ_2 (and $\Sigma_1=\sigma_1\otimes I_2$ and $\Sigma_2=I_1\otimes\sigma_2$).

Now we'll change the operator σ_1 and take the operator $\sigma_1(\theta)$ that is obtained by a rotation of σ_1 by an angle equal to θ. We will now use the fact that 35 is true for $\sigma_1(\theta)$ and σ_2 (and $\Sigma_1(\theta)=\sigma_1(\theta)\otimes I_2$ and $\Sigma_2=I_1\otimes\sigma_2$).

The eigenvectors $|\pm\rangle_\theta$ of $\sigma_1(\theta)$ are:

$$\begin{pmatrix}|+\rangle_\theta\\|-\rangle_\theta\end{pmatrix}=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}|+\rangle\\|-\rangle\end{pmatrix}$$

So we have:

$$|\Psi\rangle=c_{++}(\theta)|+_\theta+\rangle+c_{+-}(\theta)|+_\theta-\rangle+c_{-+}(\theta)|-_\theta+\rangle+c_{--}(\theta)|-_\theta-\rangle$$

Now the relation 5 is transformed into the following relation :

(8) $$|c_{++}(\theta)|^2.|c_{--}(\theta)|^2 = |c_{+-}(\theta)|^2.|c_{-+}(\theta)|^2$$

We also have

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |+\rangle_\theta \\ |-\rangle_\theta \end{pmatrix}$$

Then :
$|+\rangle = \cos\theta|+\rangle_\theta - \sin\theta|-\rangle_\theta$
$|-\rangle = \sin\theta|+\rangle_\theta + \cos\theta|-\rangle_\theta$

Because of 30, we have:

$$|\Psi\rangle = (\cos\theta.c_{++} + \sin\theta.c_{-+})| +_\theta +\rangle + (\cos\theta.c_{+-} + \sin\theta.c_{--})| +_\theta -\rangle +$$
$$(\cos\theta.c_{-+} - \sin\theta.c_{++})| -_\theta +\rangle + (\cos\theta.c_{--} - \sin\theta.c_{--})| -_\theta -\rangle$$

(9)

So:

$$c_{++}(\theta) = \cos\theta.c_{++} + \sin\theta.c_{-+}$$
$$c_{+-}(\theta) = \cos\theta.c_{+-} + \sin\theta.c_{--}$$
$$c_{-+}(\theta) = \cos\theta.c_{-+} - \sin\theta.c_{++}$$
$$c_{--}(\theta) = \cos\theta.c_{--} - \sin\theta.c_{+-}$$

We can thus write :

(10)
$$|\cos\theta.c_{++} + \sin\theta.c_{-+}|^2.|\cos\theta.c_{--} - \sin\theta.c_{+-}|^2 = |\cos\theta.c_{+-} + \sin\theta.c_{--}|^2.|\cos\theta.c_{-+} - \sin\theta.c_{++}|^2$$

All the number c_{ij} are complex. Let us write:

$$c_\mu = \rho_\mu e^{i\varphi_\mu}$$

with $\mu = (++), (+-), (-+), (--)$

We want now to rewrite 41 with these new parameters. We have:

$$|\cos\theta.c_{++} + \sin\theta.c_{-+}|^2 = (\cos\theta\rho_{++}e^{i\varphi_{++}} + \sin\theta\rho_{-+}e^{i\varphi_{-+}}).(\cos\theta\rho_{++}e^{-i\varphi_{++}} + \sin\theta\rho_{-+}e^{-i\varphi_{-+}})$$

So

$$|\cos\theta.c_{++}+\sin\theta.c_{-+}|^2=\cos\theta^2\rho_{++}^2+\sin\theta^2\rho_{-+}^2+2\cos\theta\sin\theta\rho_{++}\rho_{-+}\cos(\varphi_{++}-\varphi_{-+})$$

We have other equalities of the same kind:

$$|\cos\theta.c_{+-}+\sin\theta.c_{--}|^2=\cos\theta^2\rho_{+-}^2+\sin\theta^2\rho_{--}^2+2\cos\theta\sin\theta\rho_{+-}\rho_{--}\cos(\varphi_{+-}-\varphi_{--})$$
$$|\cos\theta.c_{-+}-\sin\theta.c_{++}|^2=\cos\theta^2\rho_{-+}^2+\sin\theta^2\rho_{++}^2-2\cos\theta\sin\theta\rho_{-+}\rho_{++}\cos(\varphi_{-+}-\varphi_{++})$$
$$|\cos\theta.c_{--}-\sin\theta.c_{+-}|^2=\cos\theta^2\rho_{--}^2+\sin\theta^2\rho_{+-}^2-2\cos\theta\sin\theta\rho_{--}\rho_{+-}\cos(\varphi_{--}-\varphi_{+-})$$

Now we can write

$$\begin{aligned}&[\cos\theta^2\rho_{++}^2+\sin\theta^2\rho_{-+}^2+2\cos\theta\sin\theta\rho_{++}\rho_{-+}\cos(\varphi_{++}-\varphi_{-+})].\\&\qquad[\cos\theta^2\rho_{--}^2+\sin\theta^2\rho_{+-}^2-2\cos\theta\sin\theta\rho_{--}\rho_{+-}\cos(\varphi_{--}-\varphi_{+-})]=\\&\qquad[\cos\theta^2\rho_{+-}^2+\sin\theta^2\rho_{--}^2+2\cos\theta\sin\theta\rho_{+-}\rho_{--}\cos(\varphi_{+-}-\varphi_{--})].\\&\qquad[\cos\theta^2\rho_{-+}^2+\sin\theta^2\rho_{++}^2-2\cos\theta\sin\theta\rho_{-+}\rho_{++}\cos(\varphi_{-+}-\varphi_{++})]\end{aligned}$$

The relation 5 gives us: $\rho_{++}\rho{--}=\rho_{+-}\rho_{-+}$. In order to simplify the formulation, we'll define the parameter k:

$$k=\frac{\rho_{++}}{\rho_{-+}}=\frac{\rho_{+-}}{\rho_{--}}$$

Then we have:

$$\begin{aligned}&[\cos\theta^2k^2+\sin\theta^2+2\cos\theta\sin\theta k\cos(\varphi_{++}-\varphi_{-+})].\\&\qquad[\cos\theta^2+\sin\theta^2k^2-2\cos\theta\sin\theta k\cos(\varphi_{--}-\varphi_{+-})]=\\&\qquad[\cos\theta^2k^2+\sin\theta^2+2\cos\theta\sin\theta k\cos(\varphi_{+-}-\varphi_{--})].\\&\qquad[\cos\theta^2+\sin\theta^2k^2-2\cos\theta\sin\theta k\cos(\varphi_{-+}-\varphi_{++})]\end{aligned}$$

After developing and simplifying we have:

$$(k^2+1)(\cos\theta\sin\theta)\cos(\varphi_{--}-\varphi_{+-})=(k^2+1)(\cos\theta\sin\theta)\cos(\varphi_{-+}-\varphi_{++})$$

And so:

- Either $\theta = 0[\frac{\pi}{2}]$
- Either $\cos(\varphi_{--} - \varphi_{+-}) = \cos(\varphi_{-+} - \varphi_{++})$

The result is:

- Either

$$(\varphi_{++} - \varphi_{-+}) = (\varphi_{+-} - \varphi_{--})[2\pi] \tag{11}$$

- Either

$$(\varphi_{++} - \varphi_{-+}) = -(\varphi_{+-} - \varphi_{--})[2\pi] \tag{12}$$

At this point we have $\rho_{++}\rho_{--} = \rho_{+-}\rho_{+-}$ and the relations 11 and 12. The relation 11 corresponds to the relation on phases when the state is a tensor product (that is to say:11 is a consequence of 30). We'll show now that the relation 12 is impossible.

In order to do that we use $\sigma(\omega)$ which is equal to $N\sigma_2 N^{-1}$ with N equal to :

$$\begin{pmatrix} \cos\omega & i\sin\omega \\ i\sin\omega & \cos\omega \end{pmatrix}$$

The eigenvectors $|\pm_\omega\rangle$ of $\sigma_2(\omega)$ are:

$$\begin{pmatrix} |+\rangle_\omega \\ |-\rangle_\omega \end{pmatrix} = \begin{pmatrix} \cos\omega & i\sin\omega \\ i\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$

We have two relations similar to 30 and 5:

$$|\Psi\rangle = c_{++}(\omega)|+_\omega +\rangle + c_{+-}(\omega)|+_\omega -\rangle + c_{-+}(\omega)|-_\omega +\rangle + c_{--}(\omega)|-_\omega -\rangle \tag{13}$$

And:

$$|c_{++}(\omega)|^2.|c_{--}(\omega)|^2 = |c_{+-}(\omega)|^2.|c_{-+}(\omega)|^2 \tag{14}$$

We also have:

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} \cos\omega & -i\sin\omega \\ -i\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} |+\rangle_\omega \\ |-\rangle_\omega \end{pmatrix}$$

Then:

$$|+\rangle = \cos\omega|+\rangle_\omega - i\sin\omega|-\rangle_\omega$$
$$|-\rangle = -i\sin\omega|+\rangle_\omega + \cos\omega|-\rangle_\omega$$

From 30, we can write:

$$
\begin{aligned}
|\Psi\rangle =&(\cos\omega c_{++} - i\sin\omega c_{-+})| +_\omega +\rangle + (\cos\omega c_{+-} - i\sin\omega c_{--})| +_\omega -\rangle + \\
&(\cos\omega c_{-+} - i\sin\omega c_{++})| -_\omega +\rangle + (\cos\omega c_{--} - i\sin\omega c_{+-})| -_\omega -\rangle
\end{aligned}
$$

So:

$$
\begin{aligned}
c_{++}(\omega) &= \cos\omega c_{++} - i\sin\omega c_{-+} \\
c_{+-}(\omega) &= \cos\omega c_{+-} - i\sin\omega c_{--} \\
c_{-+}(\omega) &= \cos\omega c_{-+} - i\sin\omega c_{++} \\
c_{--}(\omega) &= \cos\omega c_{--} - i\sin\omega c_{+-}
\end{aligned}
$$

From these we can write:

(15)

$$
|\cos\omega c_{++} - i\sin\omega c_{-+}|^2.|\cos\omega c_{--} - i\sin\omega c_{+-}|^2 = |\cos\omega c_{+-} - i\sin\omega c_{--}|^2.|\cos\omega c_{-+} - i\sin\omega c_{++}|^2
$$

We also have:

$$
\begin{aligned}
|\cos\omega c_{++} - i\sin\omega c_{-+}|^2 = & \\
&= \cos^2\omega\rho_{++}^2 + \sin^2\omega\rho_{-+}^2 + 2\cos\omega\sin\omega\rho_{++}\rho_{-+}\cos(\varphi_{++} - (\varphi_{-+} - \frac{\pi}{2})) \\
&= \cos^2\omega\rho_{++}^2 + \sin^2\omega\rho_{-+}^2 + 2\cos\omega\sin\omega\rho_{++}\rho_{-+}\cos(\varphi_{++} - \varphi_{-+} + \frac{\pi}{2}) \\
&= \cos^2\omega\rho_{++}^2 + \sin^2\omega\rho_{-+}^2 - 2\cos\omega\sin\omega\rho_{++}\rho_{-+}\sin(\varphi_{++} - \varphi_{-+})
\end{aligned}
$$

(16)

We can also write :

$$
|\cos\omega c_{--} - i\sin\omega c_{+-}|^2 = \cos^2\omega\rho_{--}^2 + \sin^2\omega\rho_{+-}^2 - 2\cos\omega\sin\omega\rho_{--}\rho_{+-}\sin(\varphi_{--} - \varphi_{+-})
$$

$$
|\cos\omega c_{+-} - i\sin\omega c_{--}|^2 = \cos^2\omega\rho_{+-}^2 + \sin^2\omega\rho_{--}^2 - 2\cos\omega\sin\omega\rho_{+-}\rho_{--}\sin(\varphi_{+-} - \varphi_{--})
$$

$$
|\cos\omega c_{-+} - i\sin\omega c_{++}|^2 = \cos^2\omega\rho_{-+}^2 + \sin^2\omega\rho_{++}^2 - 2\cos\omega\sin\omega\rho_{++}\rho_{-+}\sin(\varphi_{-+} - \varphi_{++})
$$

From these relations we can write

$$[\cos^2\omega\rho_{++}^2+\sin^2\omega\rho_{-+}^2-2\cos\omega\sin\omega\rho_{++}\rho_{-+}\sin(\varphi_{++}-\varphi_{-+})].$$
$$[\cos^2\omega\rho_{--}^2+\sin^2\omega\rho_{+-}^2-2\cos\omega\sin\omega\rho_{--}\rho_{+-}\sin(\varphi_{--}-\varphi_{+-})]=$$
$$[\cos^2\omega\rho_{+-}^2+\sin^2\omega\rho_{--}^2-2\cos\omega\sin\omega\rho_{+-}\rho_{--}\sin(\varphi_{+-}-\varphi_{--})].$$
$$[\cos^2\omega\rho_{-+}^2+\sin^2\omega\rho_{++}^2-2\cos\omega\sin\omega\rho_{++}\rho_{-+}\sin(\varphi_{-+}-\varphi_{++})]$$

With the parameters k, we obtain:

$$[\cos^2\omega k^2+\sin^2\omega-2\cos\omega\sin\omega k\sin(\varphi_{++}-\varphi_{-+})].$$
$$[\cos^2\omega+\sin^2\omega k^2-2\cos\omega\sin\omega k\sin(\varphi_{--}-\varphi_{+-})]=$$
$$[\cos^2\omega+\sin^2\omega k^2-2\cos\omega\sin\omega k\sin(\varphi_{+-}-\varphi_{--})].$$
$$[\cos^2\omega k^2+\sin^2\omega-2\cos\omega\sin\omega k\sin(\varphi_{-+}-\varphi_{++})]$$

After developing and simplifying we have:

$$(k^2+1)(\cos\omega\sin\omega)\sin(\varphi_{++}-\varphi_{-+})=(k^2+1)(\cos\omega\sin\omega)\sin(\varphi_{+-}-\varphi_{--})$$

which is equivalent to

- Either $\omega=0[\frac{\pi}{2}]$
- Either $\sin(\varphi_{++}-\varphi_{-+})=\sin(\varphi_{--}-\varphi_{+-})$

From this we get these two relations:

- Either

$$(\varphi_{++}-\varphi_{-+})=(\varphi_{+-}-\varphi_{--})[2\pi] \tag{17}$$

- Either

$$(\varphi_{++}-\varphi_{-+})=\pi-(\varphi_{+-}-\varphi_{--})[2\pi] \tag{18}$$

It's easy now to see that

- 11 is equal to 17
- the relation 12 is not compatible neither with 17 nor 18. So the relation 12 is impossible.

Finally we get this conclusion : all the numbers C_{ij} that satisfy the Condition 2 satisfy also the factorizing condition 30. We can conclude that all the state vector that satisfy the Condition 2 are in factorized state, that is to say that they are of the form:

$$|\Psi\rangle = |u\rangle \otimes |v\rangle$$

with $|u\rangle$ and $|v\rangle$ two vectors of (resp.) H_1 and H_2.

It is obvious that all the tensor product of two vectors of the Hilbert spaces H_1 and H_2 verifies B.1.

Now we can conclude : $|\Psi\rangle$ satisfies Condition 2 $\Leftrightarrow$ $|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |u\rangle \otimes |v\rangle$ with $|u\rangle$ and $|v\rangle$ two vectors of (resp.) H_1 et H_2.

B.2. **Corollary.** Condition 2 is :
(19)
$$\forall(\Sigma_1, \Sigma_2), \forall(\lambda_1, \lambda_2^{'}), P(\Sigma_1 = \lambda) = P_{\Sigma_2=\lambda'}(\Sigma_1 = \lambda), P(\Sigma_2 = \lambda^{'}) = P_{\Sigma_1=\lambda_1}(\Sigma_2 = \lambda^{'})$$

Let us now consider the relation:
(20)
$$\exists(\Sigma_1, \Sigma_2), \forall(\lambda_1, \lambda_2^{'}), P(\Sigma_1 = \lambda) = P_{\Sigma_2=\lambda'}(\Sigma_1 = \lambda), P(\Sigma_2 = \lambda^{'}) = P_{\Sigma_1=\lambda_1}(\Sigma_2 = \lambda^{'})$$

The previous demonstration shows us that if relation 20 is verified by three different couples of operators that are well chosen, then 20 is verified by all couples of operators, that is to say: then Condition 2 is verified.

Appendix C. Generalizations

In Appendix A and B we've worked on 2×2 dimension. So we have not yet given the demonstration of the mathematical results for the general case (that is to say : with N sub-subsystems in Hilbert spaces of any dimension). The generalization of the result concerning Condition 1 can be done without any difficulty. Thus, in the next paragraphs, we will be interested only in Condition 2 and suppose that Condition 1 is verified by systems that we try to know if they verify also Condition 2.

C.1. **Generalization to higher dimensions.** Let us suppose that a system S (defined in H) can be divided into 2 sub-systems S_1 and S_2 (defined (resp.) in H_1 and H_2, with $H = H_1 \otimes H_2$). In appendix B we supposed that H_1 and H_2 are 2-dimensional space. But we want to generalize to dimension n. For that, we will show the idea of the demonstration for dimension 3 and then suppose that the generalization to dimension n would be evident.

The criterion in dimension 3×3 is the same as in section 5.1 (when we worked in dimension 2×2).

Criterion of decomposition in dimension 3×3 :

The system S that is described by a state vector in H is a system composed of the 2 sub-systems S_1 and S_2 which are supposed to live respectively in the 3-dimensional Hilbert spaces H_1 and H_2 if and only if:

- *Condition 1: it is possible to make a Σ_1-measurement of S and a Σ_2-measurement of S.*
- *Condition 2:* $\forall(\Sigma_1, \Sigma_2)$, $\forall(\lambda, \lambda')$

$$P(\Sigma_1 = \lambda) = P_{\Sigma_2=\lambda'}(\Sigma_1 = \lambda)$$
$$P(\Sigma_2 = \lambda') = P_{\Sigma_1=\lambda}(\Sigma_2 = \lambda')$$

The result we want to prove is:

The system S that is described by a state vector in a 9-dimension Hilbert space H is a system composed of the two sub-systems S_1 and S_2 which are supposed to live respectively in the 3-dimension Hilbert spaces H_1 and H_2 such that $H = H_1 \otimes H_2$

$$\Updownarrow$$

$|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |u\rangle \otimes |v\rangle$ with $|u\rangle$ and $|v\rangle$ two vectors of (resp.) H_1 and H_2.

The demonstration here (in dimension 3×3) will be the same that the one in Appendix B (in dimension 2×2). That's why we start as in Appendix B, by supposing that σ_1 and σ_2 are two hermitian operators belonging (resp.) to H_1 and H_2. We note $|+\rangle$, $|0\rangle$, and $|-\rangle$ their eigenvectors. Each triplet of eigenvectors is an orthonormal basis. All vector $|\Psi\rangle$ in H can be written this way:

$$\begin{aligned} |\Psi\rangle = & C_{++}|++\rangle + C_{+0}|+0\rangle + C_{+-}|+-\rangle + \\ & C_{0+}|0+\rangle + C_{00}|00\rangle + C_{0-}|0-\rangle + \\ & C_{-+}|-+\rangle + C_{-0}|-0\rangle + C_{--}|--\rangle \end{aligned} \tag{21}$$

and:

$$|\Psi\rangle = \sum_{i,m} C_{im}|im\rangle \tag{22}$$

With this notation, in dimension 3×3, the condition for $|\Psi\rangle$ to be factorized is:

$$\forall i, \forall j (i \neq j), \forall m, \forall n (m \neq n), C_{i,m}.C_{j,n} = C_{i,n}.C_{j,m} \tag{23}$$

With this notation, Condition 2 can be written in the following way:

$$P(\Sigma_1 = +) = P_{\Sigma_2=+}(\Sigma_1 = +) = P_{\Sigma_2=0}(\Sigma_1 = +) = P_{\Sigma_2=-}(\Sigma_1 = +) \tag{24}$$

$$P(\Sigma_1 = 0) = P_{\Sigma_2=+}(\Sigma_1 = 0) = P_{\Sigma_2=0}(\Sigma_1 = 0) = P_{\Sigma_2=-}(\Sigma_1 = 0) \tag{25}$$

$$P(\Sigma_1 = -) = P_{\Sigma_2=+}(\Sigma_1 = -) = P_{\Sigma_2=0}(\Sigma_1 = -) = P_{\Sigma_2=-}(\Sigma_1 = -) \tag{26}$$

$$P(\Sigma_2 = +) = P_{\Sigma_1=+}(\Sigma_2 = +) = P_{\Sigma_1=0}(\Sigma_2 = +) = P_{\Sigma_1=-}(\Sigma_2 = +) \tag{27}$$

$$P(\Sigma_2 = 0) = P_{\Sigma_1=+}(\Sigma_2 = 0) = P_{\Sigma_1=0}(\Sigma_2 = 0) = P_{\Sigma_1=-}(\Sigma_2 = 0) \tag{28}$$

$$P(\Sigma_2 = -) = P_{\Sigma_1=+}(\Sigma_2 = -) = P_{\Sigma_1=0}(\Sigma_2 = -) = P_{\Sigma_1=-}(\Sigma_2 = -) \tag{29}$$

So our demonstration will be achieved if and only if we show this relation

$|\Psi\rangle$ *verifies Condition 2* $\Leftrightarrow$ *Equation 23 is true*

The relation "$|\Psi\rangle$ *verifies Condition 2* $\Leftarrow$ *Equation 23 is true*" is easy. So we will now suppose that $|\Psi\rangle$ verifies Condition 2 and will try to show that Equation 23 is true.

In appendix B, we had:

$$|\Psi\rangle = c_{++}|++\rangle + c_{+-}|+-\rangle + c_{-+}|-+\rangle + c_{--}|--\rangle$$

and the condition on the c_{ij} for $|\Psi\rangle$ to be a tensor product was:

$$c_{++}c_{--} = c_{+-}c_{-+}$$

So the difference consists in the fact that

- in Appendix B we had 4 coefficients c_{ij} and one relation $c_{++}c_{--} = c_{+-}c_{-+}$
- in Appendix C.1 we have 9 coefficients C_{ij} and 9 relations $C_{i,m}.C_{j,n} = C_{i,n}.C_{j.m}(i \neq j, m \neq n)$

Our strategy consists in showing that

- the treatment we applied in App. B to the four coefficients c_{ij} and the relation $c_{++}c_{--} = c_{+-}c_{-+}$
- can be done here to the 9 combinations of 4 coefficients $C_{ij}(i \neq j, m \neq n)$ and the relation $C_{i,m}.C_{j,n} = C_{i,n}.C_{j.m}$

We will show it in details for these 4 coefficients: C_{++} C_{+-}, C_{-+}, C_{--} and then show that we can do the same for the other coefficients.

What are the probabilities of a Σ_2-measurement if we made a Σ_1-measurement before it? The result of the Σ_1-measurement can be either "+" or "0" or "-".

- If the result is "+", the state vector is:

$$|\Psi\rangle = \frac{1}{\sqrt{|C_{++}|^2 + |C_{+0}|^2 + |C_{+-}|^2}}|+\rangle \otimes [C_{++}|+\rangle + C_{+0}|0\rangle + C_{+-}|-\rangle]$$

- If the result is "0" the state vector is:

$$|\Psi\rangle = \frac{1}{\sqrt{|C_{0+}|^2 + |C_{00}|^2 + |C_{0-}|^2}}|0\rangle \otimes [C_{0+}|+\rangle + C_{00}|0\rangle + C_{0-}|-\rangle]$$

- If the result is "-", the state vector is:

$$|\Psi\rangle = \frac{1}{\sqrt{|C_{-+}|^2 + |C_{-0}|^2 + |C_{--}|^2}}|-\rangle \otimes [C_{-+}|+\rangle + C_{-0}|0\rangle + C_{--}|-\rangle]$$

And then:

$$P_{\Sigma_1=+}(\Sigma_2 = +) = \frac{|C_{++}|^2}{|C_{++}|^2 + |C_{+0}|^2 + |C_{+-}|^2}$$

In order to get more simplicity, we note: $|C_{++}|^2 + |C_{+0}|^2 + |C_{+-}|^2 = D_1$, and we have:

$$P_{\Sigma_1=+}(\Sigma_2 = +) = \frac{|C_{++}|^2}{D_1}$$

We also have:

$$P_{\Sigma_1=0}(\Sigma_2 = +) = \frac{|C_{0+}|^2}{|C_{0+}|^2 + |C_{00}|^2 + |C_{0-}|^2}$$

With $|C_{0+}|^2 + |C_{00}|^2 + |C_{0-}|^2 = D_2$, we have:

$$P_{\Sigma_1=+}(\Sigma_2 = +) = \frac{|C_{0+}|^2}{D_2}$$

And finally:

$$P_{\Sigma_1=-}(\Sigma_2 = +) = \frac{|C_{-+}|^2}{|C_{-+}|^2 + |C_{-0}|^2 + |C_{--}|^2} = \frac{|C_{-+}|^2}{D_3}$$

It is easy to see that we also have:

$$P_{\Sigma_1=+}(\Sigma_2 = 0) = \frac{|C_{+0}|^2}{D_1}$$

$$P_{\Sigma_1=0}(\Sigma_2 = 0) = \frac{|C_{00}|^2}{D_2}$$

$$P_{\Sigma_1=-}(\Sigma_2 = 0) = \frac{|C_{-0}|^2}{D_3}$$

and

$$P_{\Sigma_1=+}(\Sigma_2 = -) = \frac{|C_{+-}|^2}{D_1}$$

$$P_{\Sigma_1=0}(\Sigma_2 = -) = \frac{|C_{0-}|^2}{D_2}$$

$$P_{\Sigma_1=-}(\Sigma_2 = -) = \frac{|C_{--}|^2}{D_3}$$

From equation 27 we have:

$$\frac{|C_{++}|^2}{D_1} = \frac{|C_{0+}|^2}{D_2} = \frac{|C_{-+}|^2}{D_3} \tag{30}$$

From equation 28 we have:

$$\frac{|C_{+0}|^2}{D_1} = \frac{|C_{00}|^2}{D_2} = \frac{|C_{-0}|^2}{D_3} \tag{31}$$

From equation 29 we have

$$\frac{|C_{+-}|^2}{D_1} = \frac{|C_{0-}|^2}{D_2} = \frac{|C_{--}|^2}{D_3} \tag{32}$$

From 30 and 31, we get:

$$|C_{++}|.|C_{00}| = |C_{+0}|.|C_{0+}|$$
$$|C_{++}|.|C_{-0}| = |C_{+0}|.|C_{-+}|$$
$$|C_{0+}|.|C_{-0}| = |C_{00}|.|C_{-+}|$$

From 30 and 32, we get:

$$\begin{aligned} |C_{++}|.|C_{0-}| &= |C_{+-}|.|C_{0+}| \\ |C_{++}|.|C_{--}| &= |C_{+-}|.|C_{-+}| \\ |C_{0+}|.|C_{--}| &= |C_{0-}|.|C_{-+}| \end{aligned} \tag{33}$$

From 31 and 32, we get:

$$|C_{+0}|.|C_{0-}| = |C_{+-}|.|C_{00}| \tag{34}$$
$$|C_{+0}|.|C_{--}| = |C_{+-}|.|C_{-0}|$$
$$|C_{00}|.|C_{--}| = |C_{0-}|.|C_{-0}|$$

To sumerize, we can write:

$$\forall i, \forall j (i \neq j), \forall m, \forall n (m \neq n), |C_{i,m}|.|C_{j,n}| = |C_{i,n}|.|C_{j.m}|$$

This relation must be compared to relation 23:

$$\forall i, \forall j (i \neq j), \forall m, \forall n (m \neq n), C_{i,m}.C_{j,n} = C_{i,n}.C_{j.m}$$

This problem is exactly the same as the one encountered in Appendix B and can be solved with the same strategy. Condition 2 is :

$$\forall(\Sigma_1, \Sigma_2), \forall(\lambda, \lambda^{'}), P(\Sigma_1 = \lambda) = P_{\Sigma_2=\lambda^{'}}(\Sigma_1 = \lambda), P(\Sigma_2 = \lambda^{'}) = P_{\Sigma_1=\lambda}(\Sigma_2 = \lambda^{'})$$

But up to this point we haven't used all the implications of Condition 2. In fact we choose one couple of operators Σ_1 and Σ_2 that verify the relation

$$\forall(\lambda, \lambda^{'}), P(\Sigma_1 = \lambda) = P_{\Sigma_2=\lambda^{'}}(\Sigma_1 = \lambda), P(\Sigma_2 = \lambda^{'}) = P_{\Sigma_1=\lambda}(\Sigma_2 = \lambda^{'}) \tag{35}$$

But if Condition 2 is true, the relation 35 is supposed to be true with every hermitian operators σ_1 and σ_2 (and $\Sigma_1 = \sigma_1 \otimes I_2$ and $\Sigma_2 = I_1 \otimes \sigma_2$).

So we can now change operator σ_1 and take the operator $\sigma_1(\theta)$ that is obtained by a linear transformation. We define the eigenvectors of $\sigma_1(\theta)$, with this relation:

$$\begin{pmatrix} |+\rangle_\theta \\ |0\rangle_\theta \\ |-\rangle_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} |+\rangle \\ |0\rangle \\ |-\rangle \end{pmatrix}$$

We will now use the fact that 35 is true for $\sigma_1(\theta)$ and σ_2 (and $\Sigma_1(\theta) = \sigma_1(\theta) \otimes I_2$ and $\Sigma_2 = I_1 \otimes \sigma_2$). If we make exactly the same reasoning, with the couple of operators $(\Sigma(\theta)_1, \Sigma_2)$, we have:

$$\forall i, \forall j(i \neq j), \forall m, \forall n(m \neq n), |C(\theta)_{i,m}|.|C(\theta)_{j,n}| = |C(\theta)_{i,n}|.|C(\theta)_{j.m}|$$

with:

$$|\Psi\rangle = \sum_{i,m} C_{im}(\theta)|im\rangle_\theta \tag{36}$$

But we have:

$$\begin{pmatrix} |+\rangle \\ |0\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} |+\rangle_\theta \\ |0\rangle_\theta \\ |-\rangle_\theta \end{pmatrix}$$

Then :
$|+\rangle = \cos\theta|+\rangle_\theta - \sin\theta|-\rangle_\theta$
$|0\rangle = |0\rangle_\theta$
$|-\rangle = \sin\theta|+\rangle_\theta + \cos\theta|-\rangle_\theta$

Because of 22, we have:

$$\begin{aligned}|\Psi\rangle = (\cos\theta.C_{++} + \sin\theta.C_{-+})|+_\theta +\rangle +\\ (\cos\theta.C_{+0} + \sin\theta.C_{-0})|+_\theta 0\rangle +\\ (\cos\theta.C_{+-} + \sin\theta.C_{--})|+_\theta -\rangle +\\ C_{0+}|0_\theta +\rangle +\\ C_{00}|0_\theta 0\rangle +\\ C_{0-}|0_\theta -\rangle +\\ (\cos\theta.C_{-+} - \sin\theta.C_{++})|-_\theta +\rangle +\\ (\cos\theta.C_{-0} - \sin\theta.C_{+0})|-_\theta 0\rangle +\\ (\cos\theta.C_{--} - \sin\theta.C_{+-})|-_\theta -\rangle\end{aligned}$$

From 36, we can now write:

$$C_{++}(\theta) = (\cos\theta.C_{++} + \sin\theta.C_{-+}) \tag{37}$$

$$C_{+0}(\theta) = (\cos\theta.C_{+0} + \sin\theta.C_{-0})$$

$$C_{+-}(\theta) = (\cos\theta.C_{+-} + \sin\theta.C_{--}) \tag{38}$$

$$C_{0+}(\theta) = C_{0+}$$

$$C_{00}(\theta) = C_{00}$$

$$C_{0-}(\theta) = C_{0-}$$

$$C_{-+}(\theta) = (\cos\theta.C_{-+} - \sin\theta.C_{++}) \tag{39}$$

$$C_{-0}(\theta) = (\cos\theta.C_{-0} - \sin\theta.C_{+0})$$

$$C_{--}(\theta) = (\cos\theta.C_{--} - \sin\theta.C_{--}) \tag{40}$$

Here we have to compare with Appendix B. In Appendix B, we had:

(41)

$$|\cos\theta.c_{++}+\sin\theta.c_{-+}|^2.|\cos\theta.c_{--}-\sin\theta.c_{+-}|^2 = |\cos\theta.c_{+-}+\sin\theta.c_{--}|^2.|\cos\theta.c_{-+}-\sin\theta.c_{++}|^2$$

and

$$c_{++}(\theta) = \cos\theta.c_{++} + \sin\theta.c_{-+} \tag{42}$$

$$c_{+-}(\theta) = \cos\theta.c_{+-} + \sin\theta.c_{--} \tag{43}$$

$$c_{-+}(\theta) = \cos\theta.c_{-+} - \sin\theta.c_{++} \tag{44}$$

$$c_{--}(\theta) = \cos\theta.c_{--} - \sin\theta.c_{+-} \tag{45}$$

Here we have the same relations:
41 corresponds to 33
42 corresponds to 37
43 corresponds to 38
44 corresponds to 39
45 corresponds to 40
So for C_{++}, C_{+-}, C_{-+}, C_{--}, we are exactly in the same situation as with c_{++}, c_{+-}, c_{-+}, c_{--}in appendix B.

What for the other coefficients: C_{0+}, C_{0-}, C_{+0}, C_{-0}, and C_{00}? For reason of symetry, it must be possible to treat them as we did for the 4 preceding coefficients. More precisely we can change permute "-" and "0" and make the same reasoning with:

$$\begin{pmatrix} |+\rangle_\theta \\ |0\rangle_\theta \\ |-\rangle_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |+\rangle \\ |0\rangle \\ |-\rangle \end{pmatrix}$$

With this permutation, we get the analogous of relations 41, 42, 43, 44 and 45 for the 4 coefficients C_{++} C_{+0}, C_{0+}, and C_{00}.
We get them for C_{00}, C_{0-}, C_{-0}, C_{--} with

$$\begin{pmatrix} |+\rangle_\theta \\ |0\rangle_\theta \\ |-\rangle_\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |+\rangle \\ |0\rangle \\ |-\rangle \end{pmatrix}$$

The result is: for all the C_{ij}, we are exactly in the same situation as in Appendix B for the c_{ij}.
So we can now conclude:
The system S that is described by a state vector in a 9-dimension Hilbert space H is a system composed of the two sub-systems S_1 and S_2 which are supposed to live respectively in the 3-dimension Hilbert spaces H_1 and H_2 such that $H = H_1 \otimes H_2$

$$\Updownarrow$$

$|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |u\rangle \otimes |v\rangle$ with $|u\rangle$ and $|v\rangle$ two vectors of (resp.) H_1 and H_2.
We showed:

- how we can pass from dimension 3×3 to dimension 2×2
- and then that the mathematical result in dimension 2×2 is also true in dimension 3×3

It appears also that the same method can be employed to pass from dimension $N \times N$ to dimension $(N-1) \times (N-1)$. By recurrence, we have demonstrated our theorem for any dimension $N \times N$.

C.2. **Generalization to N sub-systems.** In appendix B we worked on 2×2 dimension and we searched the condition for dividing the system S into 2 sub-systems, each sub-system living in a 2-dimensional space. Now we would like to give a proof for a system that is composed of N sub-system.

First we will work on $2 \times 2 \times 2$ dimension. It will then be easy to generalize in 2^N dimension. In 2^3 dimension, the criterion is:

The system S that is described by a state vector in H is a system composed of the 3 sub-systems S_1, S_2, and S_3 which are supposed to live respectively in the 2-dimensional Hilbert spaces H_1, H_2, and H_3 if and only if:

- *Condition 1: it is possible to make a Σ_1-measurement of S, a Σ_2-measurement of S, and a Σ_3-measurement of S.*
- *Condition 2:* $\forall(\Sigma_1, \Sigma_2, \Sigma_3)$, $\forall(\lambda, \lambda', \lambda'')$

$$P(\Sigma_1 = \lambda) = P_{\Sigma_2=\lambda'}(\Sigma_1 = \lambda) = P_{\Sigma_3=\lambda''}(\Sigma_1 = \lambda)$$
$$P(\Sigma_2 = \lambda') = P_{\Sigma_1=\lambda}(\Sigma_2 = \lambda') = P_{\Sigma_3=\lambda''}(\Sigma_2 = \lambda')$$
$$P(\Sigma_3 = \lambda'') = P_{\Sigma_1=\lambda}(\Sigma_3 = \lambda'') = P_{\Sigma_2=\lambda'}(\Sigma_3 = \lambda'')$$

The result we want to prove is:

The system S that is described by a state vector $|\Psi\rangle$ in H is a system composed of 3 sub-systems S_1, S_2, and S_3 which are supposed to live respectively in the Hilbert spaces H_1, H_2, and H_3

$$\Updownarrow$$

$|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |u\rangle \otimes |v\rangle \otimes |w\rangle$ with $|u\rangle$, $|v\rangle$, and $|w\rangle$ three vectors of (resp.) H_1, H_2, and H_3.

We suppose that $|\Psi\rangle$ satisfies Condition 2 in 2^3 dimension. Then we can define

- two Hilbert spaces $G_1 = H_1$ and $G_2 = H_2 \otimes H_3$,
- two hermitian operators $r_1 = \sigma_1$ and $r_2 = \sigma_2 \otimes I_{H_3}$,
- the corresponding hermitian operators $R_1 = r_1 \otimes I_{G_2}$ and $R_2 = I_{G_1} \otimes r_2$

Then we have

- $R_1 = \sigma_1 \otimes I_{H_2} \otimes I_{H_3}$
- $R_2 = I_{H_1} \otimes \sigma_2 \otimes I_{H_3}$

and finally

- $R_1 = \Sigma_1$
- $R_2 = \Sigma_2$

But we also know (from Condition 2 in 2^3 dimension) that if λ, λ', are two possible results for (resp.) a Σ_1-measurement and a Σ_2-measurement, then we have $P(\Sigma_1 = \lambda) = P_{\Sigma_2=\lambda'}(\Sigma_1 = \lambda)$ and $P(\Sigma_2 = \lambda') = P_{\Sigma_1=\lambda}(\Sigma_2 = \lambda')$

We can thus write

(46)
$$\exists(R_1, R_2), \forall(\lambda, \lambda'), P(R_1 = \lambda) = P_{R_2=\lambda'}(R_1 = \lambda), P(R_2 = \lambda') = P_{R_1=\lambda}(R_2 = \lambda')$$

We have to compare relation 46 to Condition 2 in 2^2 dimension that is:

(47)
$$\forall(R_1, R_2), \forall(\lambda, \lambda'), P(R_1 = \lambda) = P_{R_2=\lambda'}(R_1 = \lambda), P(R_2 = \lambda') = P_{R_1=\lambda}(R_2 = \lambda')$$

In order to demonstrate that $|\Psi\rangle$ verifies 47, we only need to find two others couples of operators that verify 46 (as seen in Appendix B, corollary B.2).

In order to find a first other couple of operators that verifies 46, we will change the operator r_1 and take the operator $r_1(\theta)$. As in Appendix B, we just have to take an operator $\sigma_1(\theta)$ in H_1 that is obtained by a rotation of σ_1 by an angle equal to θ.

The eigenvectors $|\pm\rangle_\theta$ of $\sigma_1(\theta)$ are:

$$\begin{pmatrix} |+\rangle_\theta \\ |-\rangle_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}$$

Now we can define

- the hermitian operator $r_1(\theta) = \sigma_1(\theta)$ ($r_2 = \sigma_2 \otimes I_{H_3}$ has not changed),
- the corresponding hermitian operator $R_1(\theta) = r_1(\theta) \otimes I_{G_2}$ ($R_2 = I_{G_1} \otimes r_2$ has not changed)

Then we have

- $R_1(\theta) = \sigma_1(\theta) \otimes I_{H_2} \otimes I_{H_3}$
- and still $R_2 = I_{H_1} \otimes r_2 \otimes I_{H_3}$

and finally

- $R_1(\theta) = \Sigma_1(\theta)$
- and still $R_2 = \Sigma_2$

But we know (from condition 2 in 2*2 dimension) that if μ, μ', are two possible results for (resp.) a $\Sigma_1(\theta)$-measurement and a Σ_2-measurement, then we have $P(\Sigma_1(\theta) = \mu) = P_{\Sigma_2=\mu'}(\Sigma_1(\theta) = \mu)$ and $P(\Sigma_2 = \mu') = P_{\Sigma_1(\theta)=\mu}(\Sigma_2 = \mu')$

We can thus write

(48)
$$\exists(R_1(\theta), R_2), \forall(\mu, \mu'), P(R_1(\theta) = \mu) = P_{R_2=\mu'}(R_1(\theta) = \mu), P(R_2 = \mu') = P_{R_1(\theta)=\mu}(R_2 = \mu')$$

This result means that 46 is verified by an other couple of operators. In order to demonstrate that $|\Psi\rangle$ verifies 47 we only need to find one more couple of operators that verifies 46. That's why we will now change the operator r_2 and take the operator $r_2(\omega)$. In order to do that we just have to take an operator $\sigma_2(\omega)$ in H_2 that is obtained by a rotation of σ_2 by an angle equal to ω. Then we can define

- the hermitian operator $r_2(\omega) = \sigma_2(\omega) \otimes I_{H_3}$ ($r_1 = \sigma_1$ has not changed),
- the corresponding hermitian operator $R_2(\omega) = I_{G_1} \otimes r_2(\omega)$ ($R_1 = r_1 \otimes I_{G_2}$ has not changed)

Then we have

- still $R_1 = \sigma_1 \otimes I_{H_2} \otimes I_{H_3}$
- but now $R_2 = I_{H_1} \otimes r_2(\omega) \otimes I_{H_3}$

and finally

- still $R_1 = \Sigma_1$
- and $R_2 = \Sigma_2(\omega)$

So we can write:

(49)
$$\exists(R_1, R_2(\theta)), \forall(\nu, \nu'), P(R_1 = \nu) = P_{R_2(\theta)=\nu'}(R_1 = \nu), P(R_2(\theta) = \nu') = P_{R_1=\nu}(R_2(\theta) = \nu')$$

From 46, 48, and 49, we can now say that $|\Psi\rangle$ verifies Condition 2.
Thus the result is:

The system S that is described by a state vector $|\Psi\rangle$ in H is composed of 3 sub-systems S_1, S_2, and S_3 living (resp.) in three 2-dimensional Hilbert spaces H_1, H_2 and H_3 with $H = H_1 \otimes H_2 \otimes H_3$

$$\Downarrow$$

S is composed of 2 sub-systems P_1, P_2 living (resp.) in two Hilbert spaces $G_1 = H_1$ and $G_2 = H_2 \otimes H_3$

But we can use the result of Appendix B

$|\Psi\rangle$, a vector of H, satisfies Condition 2 (in 2^2 dimension) $\Leftrightarrow$ $|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |u\rangle \otimes |u'\rangle$ with $|u\rangle$ and $|u'\rangle$ two vectors of (resp.) G_1 and G_2 with $H = G_1 \otimes G_2$.

If we apply this result to $|\Psi\rangle$ we have

The system S that is described by a state vector $|\Psi\rangle$ in H is composed of 3 sub-systems S_1, S_2, and S_3 living (resp.) in three 2-dimensional Hilbert spaces H_1, H_2 and H_3 with $H = H_1 \otimes H_2 \otimes H_3$

$$\Downarrow$$

$|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |u\rangle \otimes |u'\rangle$ with $|u\rangle$ and $|u'\rangle$ two vectors of (resp.) H_1 and $H_2 \otimes H_3$.

It's easy to see that the reciprocal is true. So we finally have:

The system S that is described by a state vector $|\Psi\rangle$ in H is composed of 3 sub-systems S_1, S_2, and S_3 living (resp.) in three 2-dimensional Hilbert spaces H_1, H_2 and H_3 with $H = H_1 \otimes H_2 \otimes H_3$

$$\Updownarrow$$

$|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |u\rangle \otimes |u'\rangle$ with $|u\rangle$ and $|u'\rangle$ two vectors of (resp.) H_1 and $H_2 \otimes H_3$.

For reason of symetry it is easy to see that we can also write:

The system S that is described by a state vector $|\Psi\rangle$ in H is composed of 3 sub-systems S_1, S_2, and S_3 living (resp.) in three 2-dimensional Hilbert spaces H_1, H_2 and H_3 with $H = H_1 \otimes H_2 \otimes H_3$

$$\Updownarrow$$

$|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |v\rangle \otimes |v'\rangle$ with $|v\rangle$ and $|v'\rangle$ two vectors of (resp.) H_2 and $H_1 \otimes H_3$.

$$\Updownarrow$$

$|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |w\rangle \otimes |w'\rangle$ with $|w\rangle$ and $|w'\rangle$ two vectors of (resp.) H_3 and $H_1 \otimes H_2$.

From these we can deduce

The system S that is described by a state vector $|\Psi\rangle$ in H is composed of 3 sub-systems S_1, S_2, and S_3 living (resp.) in three Hilbert spaces H_1, H_2 and H_3 with $H = H_1 \otimes H_2 \otimes H_3$

$$\Updownarrow$$

$|\Psi\rangle$ is a tensor product of the form $|\Psi\rangle = |u\rangle \otimes |m\rangle \otimes |i\rangle$ with $|u\rangle$, $|m\rangle$ and $|i\rangle$ three vectors of (resp.) H_1, H_1 and H_3.

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